

## RESEARCH ARTICLE

### $L^p$ solutions of finite and infinite time interval BSDEs with non-Lipschitz coefficients

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In this paper, we are interested in solving multidimensional backward stochastic differential equations (BSDEs) in  $L^p$  ( $p > 1$ ) under weaker assumptions on the coefficients, considering both a finite and an infinite time interval. We establish a general existence and uniqueness result of solutions in  $L^p$  ( $p > 1$ ) to finite and infinite time interval BSDEs with non-Lipschitz coefficients, which includes the corresponding results in Pardoux and Peng [11], Mao [9], Chen [4], Constantin [6], Wang and Wang [13], Chen and Wang [5] and Wang and Huang [12] as its particular cases.

**Keywords:** Backward stochastic differential equation; Infinite time interval; Non-Lipschitz coefficients; Mao's condition;  $L^p$  solution; Existence and uniqueness

**AMS Subject Classification:** primary 60H10

#### 1. Introduction

In this paper, we consider the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s) \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T], \quad (1.1)$$

where the time horizon  $T > 0$  is a finite or infinite constant, the terminal condition  $\xi$  is a  $k$ -dimensional random variable, the generator  $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k \times d} \rightarrow \mathbf{R}^k$  is progressively measurable for each  $(y, z)$ , and  $B$  is a  $d$ -dimensional Brownian motion. The solution  $(y, z)$  is a pair of adapted processes. The  $(\xi, T, g)$  describe the coefficients (parameters) of BSDE (1.1).

Such equations, in the nonlinear case, were first introduced by Pardoux and Peng [11], who established an existence and uniqueness result for solutions in  $L^2$  to BSDEs under the Lipschitz assumption of the generator  $g$ . Since then, BSDEs have attracted much interest, and have become an important mathematical tool in many fields including financial mathematics, stochastic games and optimal control. In particular, much effort has been made to relax the Lipschitz hypothesis on  $g$ , for instance, in the one dimensional setting ( $k=1$ ), Lepeltier and San Martin [8] have proved the existence of a solution in  $L^2$  for BSDE (1.1) when  $g$  is continuous and

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of linear growth in  $(y, z)$ , Kobylanski [7] obtained the existence and uniqueness of a solution in  $L^2$  when  $g$  has a quadratic growth in  $z$  and the terminal condition  $\xi$  is bounded, and then Briand and Hu [2] and Briand and Hu [3] further extended the result of Kobylanski [7] to the case of unbounded terminal conditions.

Mao [9] proposed the following non-Lipschitz assumption for the generator  $g$  of multidimensional BSDEs:

$$(H1) \quad dP \times dt - a.s., \quad \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d},$$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)|^2 \leq \kappa(|y_1 - y_2|^2) + c|z_1 - z_2|^2,$$

where  $c > 0$  and  $\kappa(\cdot)$  is a concave and nondecreasing function from  $\mathbf{R}^+$  to  $\mathbf{R}^+$  such that  $\kappa(0) = 0$ ,  $\kappa(u) > 0$  for  $u > 0$  and  $\int_{0+} \kappa^{-1}(u) du = +\infty$ . Under this assumption, he proved that BSDE (1.1) with  $0 < T < +\infty$  has a unique solution in  $L^2$ .

Wang and Wang [13] proposed another non-Lipschitz condition for the generator  $g$  of multidimensional BSDEs and Wang and Huang [12] further generalized it as follows:

$$(H2) \quad dP \times dt - a.s., \quad \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d},$$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)|^2 \leq \kappa(t, |y_1 - y_2|^2) + c|z_1 - z_2|^2,$$

where  $c > 1$  and  $\kappa(\cdot, \cdot) \in \mathbf{S}[T, a(\cdot), b(\cdot)]$ , here and henceforth for  $0 < T \leq +\infty$ ,  $\mathbf{S}[T, a(\cdot), b(\cdot)]$  denotes the set of functions  $\kappa(\cdot, \cdot) : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}^+$  satisfying the following two conditions:

- For fixed  $t \in [0, T]$ ,  $\kappa(t, \cdot)$  is a continuous, concave and nondecreasing function with  $\kappa(t, 0) = 0$ , and for each  $t \in [0, T]$ ,  $\kappa(t, u) \leq a(t) + b(t)u$ , where the functions  $a(\cdot), b(\cdot) : [0, T] \mapsto \mathbf{R}^+$  satisfy  $\int_0^T [a(t) + b(t)] dt < +\infty$ ;
- The following ODE,  $u'(t) = -\kappa(t, u), t \in [0, T]$  with  $u(T) = 0$ , has a unique solution  $u(t) = 0, t \in [0, T]$ .

Under (H2), they proved the existence and uniqueness of the solution in  $L^2$  to BSDE (1.1) with  $0 < T < +\infty$  and, with the help of Bihari's inequality, they proved that their result includes that of Mao [9].

Moreover, Chen [4] and Chen and Wang [5] proposed the following non-uniformly Lipschitz condition for the generator  $g$  of multidimensional BSDEs:

$$(H3) \quad dP \times dt - a.s., \quad \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d},$$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq u(t)|y_1 - y_2| + v(t)|z_1 - z_2|,$$

where  $u(\cdot), v(\cdot) : [0, T] \mapsto \mathbf{R}^+$  satisfy  $\int_0^T u(t) dt < +\infty$  and  $\int_0^T v^2(t) dt < +\infty$ .

Under (H3), they established the existence and uniqueness of the solution in  $L^2$  to BSDE (1.1) with  $0 < T \leq +\infty$ .

Furthermore, in the case where  $0 < T < +\infty$ , Pardoux [10] established the existence and uniqueness result of a solution in  $L^2$  for BSDE (1.1) where  $g$  satisfies the particular monotonicity condition in  $y$ . Using the same monotonicity condition for  $g$ , Briand et al. [1] investigated the existence and uniqueness of a solution in  $L^p$  ( $p > 1$ ) for BSDE (1.1).

In this paper, we are interested in solving BSDEs in  $L^p$  ( $p > 1$ ) under weaker

assumptions on the coefficients, considering both a finite and an infinite time interval. We establish a general existence and uniqueness theorem of solutions in  $L^p$  ( $p > 1$ ) to finite time and infinite interval BSDEs (see Theorem 3.2 in Section 3), which includes the corresponding results in Pardoux and Peng [11], Mao [9], Chen [4], Constantin [6], Wang and Wang [13], Chen and Wang [5] and Wang and Huang [12] as its particular cases. The paper is organized as follows. We introduce some preliminaries and lemmas in Section 2 and put forward and prove our main results in Section 3. Some examples, corollaries and remarks are given in Section 4 to show that Theorem 3.2 of this paper is a generalization of some results mentioned above. Finally, some further discussions with respect to our main result are provided in Section 4.

## 2. Preliminaries and Lemmas

Let us first introduce some notation. First of all, let us fix two real numbers  $0 < T \leq +\infty$  and  $p > 1$ , and two positive integers  $k$  and  $d$ . Let  $(\Omega, \mathcal{F}, P)$  be a probability space carrying a standard  $d$ -dimensional Brownian motion  $(B_t)_{t \geq 0}$ . Let  $(\mathcal{F}_t)_{t \geq 0}$  be the natural  $\sigma$ -algebra generated by  $(B_t)_{t \geq 0}$  and  $\mathcal{F} = \mathcal{F}_T$ . In this paper, the Euclidean norm of a vector  $y \in \mathbf{R}^k$  will be defined by  $|y|$ , and for a  $k \times d$  matrix  $z$ , we define  $|z| = \sqrt{\text{Tr}(zz^*)}$ , where  $z^*$  is the transpose of  $z$ . Let  $a \wedge b$  represent the minimum of  $a, b \in \mathbf{R}$  and  $\langle x, y \rangle$  the inner product of  $x, y \in \mathbf{R}^k$ . We denote by  $L^p(\mathbf{R}^k)$  the set of all  $\mathbf{R}^k$ -valued and  $\mathcal{F}_T$ -measurable random vectors  $\xi$  such that  $\mathbf{E}[|\xi|^p] < +\infty$ . Let  $\mathcal{S}^p(0, T; \mathbf{R}^k)$  denote the set of  $\mathbf{R}^k$ -valued, adapted and continuous processes  $(Y_t)_{t \in [0, T]}$  such that

$$\|Y\|_{\mathcal{S}^p} := \left\{ \mathbf{E} \left[ \sup_{t \in [0, T]} |Y_t|^p \right] \right\}^{1/p} < +\infty.$$

Moreover, let  $M^p(0, T; \mathbf{R}^k)$  (resp.  $M^p(0, T; \mathbf{R}^{k \times d})$ ) denote the set of  $(\mathcal{F}_t)$ -progressively measurable  $\mathbf{R}^k$ -valued ( $\mathbf{R}^{k \times d}$ -valued) processes  $(Z_t)_{t \in [0, T]}$  such that

$$\|Z\|_{M^p} := \left\{ \mathbf{E} \left[ \left( \int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{1/p} < +\infty.$$

Obviously, both  $\mathcal{S}^p$  and  $M^p$  are Banach spaces. As mentioned in the introduction, we will deal only with BSDEs which are equations of type (1.1), where the terminal condition  $\xi$  belongs to the space  $L^p(\mathbf{R}^k)$ , and the generator  $g$  is  $(\mathcal{F}_t)$ -progressively measurable for each  $(y, z)$ .

**Definition 2.1:** A pair of processes  $(y_t, z_t)_{t \in [0, T]}$  is called a solution in  $L^p$  to BSDE (1.1), if  $(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}^p(0, T; \mathbf{R}^k) \times M^p(0, T; \mathbf{R}^{k \times d})$  and satisfies (1.1).

Let us introduce the following “Backward Gronwall Inequality”. We omit the standard proof.

**Lemma 2.2:** Let  $0 < T \leq \infty$ ,  $\alpha(t) : [0, T] \mapsto \mathbf{R}^+$  be a decreasing function,  $\beta(t) : [0, T] \mapsto \mathbf{R}^+$  satisfy  $\int_0^T \beta(s) ds < +\infty$ , and  $u(t) : [0, T] \mapsto \mathbf{R}^+$  be a continuous function with  $\sup_{t \in [0, T]} u(t) < +\infty$  such that

$$u(t) \leq \alpha(t) + \int_t^T \beta(s)u(s) ds, \quad t \in [0, T].$$

Then we have

$$u(t) \leq \alpha(t) e^{\int_t^T \beta(s) \, ds}, \quad t \in [0, T].$$

The following Lemma 2.3 comes from Corollary 2.3 of Briand et al. [1], which is the starting point of this paper.

**Lemma 2.3:** *If  $(y_t, z_t)_{t \in [0, T]}$  be a solution in  $L^p$  of BSDE (1.1),  $c(p) = p/2[(p-1) \wedge 1]$  and  $0 \leq t \leq T$ , then*

$$\begin{aligned} |y_t|^p + c(p) \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds &\leq |\xi|^p + p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, g(s, y_s, z_s) \rangle \, ds \\ &\quad - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle. \end{aligned}$$

Now we establish the following two propositions important in the proof of our main result. In stating these propositions it will be useful to introduce the following assumption on the generator  $g$ :

$$(A) \quad dP \times dt - a.s., \forall (y, z) \in \mathbf{R}^k \times \mathbf{R}^{k \times d},$$

$$|g(\omega, t, y, z)| \leq \mu(t) \left[ \psi^{\frac{1}{p}}(t, |y|^p) + \varphi_t \right] + \nu(t)|z| + f_t,$$

where  $\mu(\cdot), \nu(\cdot) : [0, T] \mapsto \mathbf{R}^+$  with  $\int_0^T [\mu^{\frac{p}{p-1}}(t) + \nu^2(t)] \, dt < +\infty$ , both  $\varphi_t$  and  $f_t$  are nonnegative,  $(\mathcal{F}_t)$ -progressively measurable processes with  $\mathbf{E} \left[ \int_0^T \varphi_t^p \, dt \right] < +\infty$  and  $\mathbf{E} \left[ \left( \int_0^T f_t \, dt \right)^p \right] < +\infty$ , and  $\psi(\cdot, \cdot) \in \mathbf{S}[T, a(\cdot), b(\cdot)]$ .

**Remark 2.4:** If  $\psi(\cdot, \cdot) \in \mathbf{S}[T, a(\cdot), b(\cdot)]$  and  $y_t \in \mathcal{S}^p$ , then

$$\mathbf{E} \left[ \int_0^T \psi(t, |y_t|^p) \, dt \right] \leq \int_0^T a(t) \, dt + \int_0^T b(t) \, dt \cdot \mathbf{E} \left[ \sup_{t \in [0, T]} |y_t|^p \right] < +\infty.$$

**Proposition 2.5:** *Let assumption (A) hold and let  $(y_t, z_t)_{t \in [0, T]}$  be a solution in  $L^p$  to BSDE (1.1). Denote  $\bar{\mu}(t) = \int_t^T \mu^{\frac{p}{p-1}}(s) \, ds$  and  $\bar{\nu}(t) = \int_t^T \nu^2(s) \, ds$ . Then there exists a constant  $m_p > 0$  depending only on  $p$  such that for each  $t \in [0, T]$ ,*

$$\begin{aligned} \mathbf{E} \left[ \left( \int_t^T |z_s|^2 \, ds \right)^{p/2} \right] &\leq m_p |\xi|^p + m_p C_t \left\{ \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] + \int_t^T \psi(s, \mathbf{E}[|y_s|^p]) \, ds \right. \\ &\quad \left. + \mathbf{E} \left[ \int_t^T \varphi_s^p \, ds \right] + \mathbf{E} \left[ \left( \int_t^T f_s \, ds \right)^p \right] \right\}, \end{aligned}$$

where  $C_t := 1 + \bar{\mu}^{p-1}(t) + \bar{\mu}^{2p-2}(t) + \bar{\nu}^{p/2}(t) + \bar{\nu}^p(t)$ .

**Proof:** Applying Itô's formula to  $|y_t|^2$  yields

$$|y_t|^2 + \int_t^T |z_s|^2 \, ds = |\xi|^2 + 2 \int_t^T \langle y_s, g(s, y_s, z_s) \rangle \, ds - 2 \int_t^T \langle y_s, z_s dB_s \rangle.$$

It follows from assumption (A) that for each  $s \in [t, T]$ ,

$$\begin{aligned} 2\langle y_s, g(s, y_s, z_s) \rangle &\leq 2 \left( \sup_{s \in [t, T]} |y_s| \right) \left( \mu(s) \left[ \psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s \right] + f_s \right) \\ &\quad + 2 \left( \sup_{s \in [t, T]} |y_s|^2 \right) \cdot \nu^2(s) + \frac{|z_s|^2}{2}. \end{aligned}$$

Thus, in view of the inequality  $2ab \leq a^2 + b^2$ , we get that

$$\begin{aligned} \frac{1}{2} \int_t^T |z_s|^2 \, ds &\leq |\xi|^2 + (2 + 2\bar{\nu}(t)) \cdot \left( \sup_{s \in [t, T]} |y_s|^2 \right) + \left[ \int_t^T f_s \, ds \right]^2 \\ &\quad + \left\{ \int_t^T \mu(s) \left[ \psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s \right] \, ds \right\}^2 + 2 \left| \int_t^T \langle y_s, z_s dB_s \rangle \right|. \end{aligned}$$

It follows from Hölder's inequality and the inequality  $(a + b)^p \leq 2^p(a^p + b^p)$  that

$$\begin{aligned} \left[ \int_t^T \mu(s) \left[ \psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s \right] \, ds \right]^p &\leq \left[ \int_t^T \mu^{\frac{p}{p-1}}(s) \, ds \right]^{p-1} \cdot \int_t^T \left[ \psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s \right]^p \, ds \\ &\leq \bar{\mu}^{p-1}(t) \cdot 2^p \int_t^T [\psi(s, |y_s|^p) + \varphi_s^p] \, ds. \end{aligned}$$

Then there exists a constant  $a_p > 0$  depending only on  $p$  such that

$$\begin{aligned} \left[ \int_t^T |z_s|^2 \, ds \right]^{p/2} &\leq a_p |\xi|^p + c_t \left\{ \sup_{s \in [t, T]} |y_s|^p + \int_t^T \psi(s, |y_s|^p) \, ds + \int_t^T \varphi_s^p \, ds \right. \\ &\quad \left. + \left[ \int_t^T f_s \, ds \right]^p + \left| \int_t^T \langle y_s, z_s dB_s \rangle \right|^{p/2} \right\}, \end{aligned} \quad (2.1)$$

where  $c_t := a_p (1 + \bar{\mu}^{p-1}(t) + \bar{\nu}^{p/2}(t))$ .

Furthermore, the Burkholder–Davis–Gundy (BDG) inequality implies that there exists a constant  $d_p > 0$  depending only on  $p$  such that for each  $t \in [0, T]$ ,

$$\begin{aligned} c_t \mathbf{E} \left[ \left| \int_t^T \langle y_s, z_s dB_s \rangle \right|^{p/2} \right] &\leq c_t d_p \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^{p/2} \cdot \left( \int_t^T |z_s|^2 \, ds \right)^{p/4} \right] \\ &\leq \frac{c_t^2 d_p^2}{2} \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] + \frac{1}{2} \mathbf{E} \left[ \left( \int_t^T |z_s|^2 \, ds \right)^{p/2} \right]. \end{aligned}$$

Returning to the estimate (2.1), we get that for each  $t \in [0, T]$ ,

$$\begin{aligned} \mathbf{E} \left[ \left( \int_t^T |z_s|^2 \, ds \right)^{p/2} \right] &\leq 2a_p |\xi|^p + (2c_t + c_t^2 d_p^2) \left\{ \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] + \mathbf{E} \left[ \int_t^T \varphi_s^p \, ds \right] \right. \\ &\quad \left. + \mathbf{E} \left[ \int_t^T \psi(s, |y_s|^p) \, ds \right] + \mathbf{E} \left[ \left( \int_t^T f_s \, ds \right)^p \right] \right\}. \end{aligned}$$

Then it follows from the definition of the function  $c_t$  that there exists a constant

$b_p > 0$  depending only on  $p$  such that

$$2c_t + c_t^2 d_p^2 \leq b_p \left( 1 + \bar{\mu}^{p-1}(t) + \bar{\mu}^{2p-2}(t) + \bar{\nu}^{p/2}(t) + \bar{\nu}^p(t) \right).$$

Thus, by taking  $m_p = 2a_p + b_p$ , in view of the fact that  $\psi(s, \cdot)$  is a concave function for each  $s \in [0, T]$ , the conclusion of Proposition 2.5 follows from Fubini's theorem and Jensen's inequality, completing the proof.  $\square$

**Proposition 2.6:** *Let assumption (A) hold and let  $(y_t, z_t)_{t \in [0, T]}$  be a solution in  $L^p$  to BSDE (1.1). Denote  $\bar{\mu}(t) = \int_t^T \mu^{\frac{p}{p-1}}(s) ds$  and  $\bar{\nu}(t) = \int_t^T \nu^2(s) ds$ . Then there exists a constant  $k_p > 0$  depending only on  $p$  with  $K_t := e^{k_p(\bar{\mu}(t) + \bar{\nu}(t))}$  such that for each  $t \in [0, T]$ ,*

$$\mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] \leq K_t \left\{ k_p \mathbf{E}[|\xi|^p] + k_p \mathbf{E} \left[ \left( \int_t^T f_s ds \right)^p \right] + \frac{1}{2} \mathbf{E} \left[ \int_t^T \varphi_s^p ds \right] + \frac{1}{2} \int_t^T \psi(s, \mathbf{E}[|y_s|^p]) ds \right\}.$$

**Proof:** Assumption (A) yields that  $\langle y_s, g(s, y_s, z_s) \rangle \leq |y_s| \{ \mu(s) [\psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s] + \nu(s)|z_s| + f_s \}$ , from which and Lemma 2.3 we deduce that, with probability one, for each  $t \in [0, T]$ ,

$$\begin{aligned} |y_t|^p + c(p) \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 ds &\leq |\xi|^p - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle \\ &\quad + p \int_t^T |y_s|^{p-1} \left\{ \mu(s) \left[ \psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s \right] + \nu(s)|z_s| + f_s \right\} ds. \end{aligned}$$

From Young's inequality ( $a^r b^{1-r} \leq ra + (1-r)b$  for each  $a \geq 0, b \geq 0$  and  $0 < r < 1$ ) and the inequality  $(a+b)^p \leq 2^p(a^p + b^p)$  it follows that

$$\begin{aligned} &p \int_t^T |y_s|^{p-1} \mu(s) \left( \psi^{\frac{1}{p}}(s, |y_s|^p) + \varphi_s \right) ds \\ &\leq (p-1) \delta^{\frac{1}{p-1}} \int_t^T |y_s|^p \mu^{\frac{p}{p-1}}(s) ds + \frac{2^p}{\delta} \int_t^T (\psi(s, |y_s|^p) + \varphi_s^p) ds, \end{aligned}$$

where  $\delta > 0$  will be chosen later. Thus, by assumption (A) and Remark 2.4 we deduce first from the previous two inequalities that,  $\int_0^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 ds < +\infty, dP - a.s.$  Moreover, from the inequality  $ab \leq (a^2 + b^2)/2$  we get that

$$p\nu(s)|y_s|^{p-1}|z_s| \leq \frac{p\nu^2(s)}{1 \wedge (p-1)}|y_s|^p + \frac{c(p)}{2}|y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2.$$

Then for each  $t \in [0, T]$ , we have

$$|y_t|^p + \frac{c(p)}{2} \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 ds \leq X_t - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle, \quad (2.2)$$

where

$$X_t := |\xi|^p + d_{p,\delta} \int_t^T \left( \mu^{\frac{p}{p-1}}(s) + \nu^2(s) \right) |y_s|^p ds + \frac{2^p}{\delta} \int_t^T (\psi(s, |y_s|^p) + \varphi_s^p) ds + p \int_t^T |y_s|^{p-1} f_s ds$$

with  $d_{p,\delta} = (p-1)\delta^{1/(p-1)} + p/[1 \wedge (p-1)] > 0$ .

It follows from the BDG inequality that  $\{M_t := \int_0^t |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle\}_{t \in [0, T]}$  is a uniformly integrable martingale. In fact, by Young's inequality we have

$$\begin{aligned} \mathbf{E} \left[ \langle M, M \rangle_T^{1/2} \right] &\leq \mathbf{E} \left[ \sup_{s \in [0, T]} |y_s|^{p-1} \cdot \left( \int_0^T |z_s|^2 ds \right)^{1/2} \right] \\ &\leq \frac{(p-1)}{p} \mathbf{E} \left[ \sup_{s \in [0, T]} |y_s|^p \right] + \frac{1}{p} \mathbf{E} \left[ \left( \int_0^T |z_s|^2 ds \right)^{p/2} \right] < +\infty. \end{aligned}$$

Returning to inequality (2.2) and taking the expectation, we get both

$$\frac{c(p)}{2} \mathbf{E} \left[ \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 ds \right] \leq \mathbf{E}[X_t] \quad (2.3)$$

and

$$\mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] \leq \mathbf{E}[X_t] + \bar{k}_p \mathbf{E} \left[ (\langle M, M \rangle_T - \langle M, M \rangle_t)^{1/2} \right], \quad (2.4)$$

where the  $\bar{k}_p > 0$  only depends on  $p$ . The last step uses the BDG inequality.

On the other hand, Young's inequality implies that

$$\begin{aligned} &\bar{k}_p \mathbf{E} \left[ (\langle M, M \rangle_T - \langle M, M \rangle_t)^{1/2} \right] \\ &\leq \bar{k}_p \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^{p/2} \cdot \left( \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 ds \right)^{1/2} \right] \\ &\leq \frac{1}{2} \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] + \frac{\bar{k}_p^2}{2} \mathbf{E} \left[ \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 ds \right]. \end{aligned}$$

We may now combine inequalities (2.3) and (2.4) to obtain the existence of a constant  $k'_p > 0$  such that  $\mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] \leq k'_p \mathbf{E}[X_t]$ . Another application of Young's inequality yields the existence of a constant  $k''_p > 0$  depending only on  $p$  such that

$$\begin{aligned} pk'_p \mathbf{E} \left[ \int_t^T |y_s|^{p-1} f_s ds \right] &\leq pk'_p \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^{p-1} \int_t^T f_s ds \right] \\ &\leq \frac{1}{2} \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] + \frac{k''_p}{2} \mathbf{E} \left[ \left( \int_t^T f_s ds \right)^p \right]. \end{aligned}$$

Thus, using the definition of  $X_t$  we can deduce that

$$\begin{aligned} \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] &\leq 2k'_p \mathbf{E} \left[ |\xi|^p + d_{p,\delta} \int_t^T \left( \mu^{\frac{p}{p-1}}(s) + \nu^2(s) \right) |y_s|^p ds \right] \\ &\quad + 2k'_p \mathbf{E} \left[ \frac{2^p}{\delta} \int_t^T (\psi(s, |y_s|^p) + \varphi_s^p) ds \right] + k''_p \mathbf{E} \left[ \left( \int_t^T f_s ds \right)^p \right]. \end{aligned}$$

By letting  $\delta = 2^{p+2}k'_p$  and  $h_t = \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right]$  in the previous inequality and using Fubini's theorem and Jensen's inequality, noticing that  $\psi(s, \cdot)$  is a concave function for each  $s \in [0, T]$ , we know that for each  $t \in [0, T]$ ,

$$\begin{aligned} h_t &\leq 2k'_p \mathbf{E}[|\xi|^p] + k''_p \mathbf{E} \left[ \left( \int_t^T f_s \, ds \right)^p \right] + \frac{1}{2} \mathbf{E} \left[ \int_t^T \varphi_s^p \, ds \right] \\ &\quad + \frac{1}{2} \int_t^T \psi(s, \mathbf{E}[|y_s|^p]) \, ds + 2k'_p d_{p, \delta} \int_t^T \left( \mu^{\frac{p}{p-1}}(s) + \nu^2(s) \right) h_s \, ds. \end{aligned}$$

Finally, in view of assumption (A), the Backward Gronwall inequality (Lemma 2.2) yields that for each  $t \in [0, T]$ ,

$$\begin{aligned} h_t &\leq e^{2k'_p d_{p, \delta} (\bar{\mu}(t) + \bar{\nu}(t))} \left\{ 2k'_p \mathbf{E}[|\xi|^p] + k''_p \mathbf{E} \left[ \left( \int_t^T f_s \, ds \right)^p \right] \right. \\ &\quad \left. + \frac{1}{2} \mathbf{E} \left[ \int_t^T \varphi_s^p \, ds \right] + \frac{1}{2} \int_t^T \psi(s, \mathbf{E}[|y_s|^p]) \, ds \right\}. \end{aligned}$$

The proof of Proposition 2.6 is thus completed.  $\square$

### 3. Main Result and Its Proof

In this section, we will put forward and prove our main result. Let us first introduce the following assumptions, where we assume that  $0 < T \leq +\infty$ :

(H4)  $dP \times dt - a.s., \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d},$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq \alpha(t) \rho^{\frac{1}{p}}(t, |y_1 - y_2|^p) + \beta(t) |z_1 - z_2|,$$

where  $\alpha(\cdot), \beta(\cdot) : [0, T] \mapsto \mathbf{R}^+$  satisfy the condition  $\int_0^T \left( \alpha^{\frac{p}{p-1}}(t) + \beta^2(t) \right) dt < +\infty$  and the function  $\rho(\cdot, \cdot)$  belongs to  $\mathbf{S}[T, a(\cdot), b(\cdot)]$ ;

$$(H5) \quad \mathbf{E} \left[ \left( \int_0^T |g(t, 0, 0)| \, dt \right)^p \right] < +\infty.$$

**Remark 3.1:** It follows from Young's inequality, the inequality  $(a+b)^p \leq 2^p(a^p + b^p)$ , and assumption (H4) that

$$\int_0^T \left[ \alpha(t) \left( a^{\frac{1}{p}}(t) + b^{\frac{1}{p}}(t) \right) \right] dt \leq \int_0^T \left[ \frac{p-1}{p} \alpha^{\frac{p}{p-1}}(t) + \frac{2^p}{p} (a(t) + b(t)) \right] dt < +\infty.$$

Furthermore, Hölder's inequality yields that  $g(\cdot, 0, 0) \in M^p(0, T; \mathbf{R}^k)$  implies (H5) in the case where  $T < +\infty$ .

The following Theorem 3.2 is the main result of this paper.

**Theorem 3.2:** *Let  $0 < T \leq +\infty$  and  $g$  satisfy (H4) and (H5). Then, for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .*

In order to prove Theorem 3.2, we need first to establish the following Proposition 3.3, which is just Theorem 1.2 of Chen and Wang [5] when  $p = 2$ .

**Proposition 3.3:** *Let  $0 < T \leq +\infty$  and  $g$  satisfy (H3) and (H5). Then, for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .*



**Proof:** Define  $\hat{u}([t_1, t_2]) := \int_{t_1}^{t_2} u(s)ds$  and  $\hat{v}([t_1, t_2]) := \int_{t_1}^{t_2} v^2(s)ds$  for each  $0 \leq t_1 \leq t_2 \leq T$ . Assume that  $(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}^p(0, T; \mathbf{R}^k) \times M^p(0, T; \mathbf{R}^{k \times d})$ . It follows from assumption (H3) that  $|g(s, y_s, z_s)| \leq |g(s, 0, 0)| + u(s)|y_s| + v(s)|z_s|$  and then from the inequality  $(a + b + c)^p \leq 3^p(a^p + b^p + c^p)$  and Hölder's inequality that for each  $t \in [0, T]$ ,

$$\begin{aligned} \mathbf{E} \left[ \left( \int_t^T |g(s, y_s, z_s)| ds \right)^p \right] &\leq 3^p \mathbf{E} \left[ \left( \int_t^T |g(s, 0, 0)| ds \right)^p \right] + 3^p \hat{u}^p([t, T]) \cdot \mathbf{E} \left[ \sup_{s \in [0, T]} |y_s|^p \right] \\ &\quad + 3^p \hat{v}^{\frac{p}{2}}([t, T]) \cdot \mathbf{E} \left[ \left( \int_t^T |z_s|^2 ds \right)^{p/2} \right] < +\infty, \end{aligned}$$

As a result, the process  $\left\{ \mathbf{E} \left[ \xi + \int_0^T g(s, y_s, z_s) ds \middle| \mathcal{F}_t \right] \right\}_{0 \leq t \leq T}$  is an  $L^p$  martingale. It follows from the martingale representation theorem that there exists a unique process  $Z_t \in M^p(0, T; \mathbf{R}^{k \times d})$  such that

$$\mathbf{E} \left[ \xi + \int_0^T g(s, y_s, z_s) ds \middle| \mathcal{F}_t \right] = \mathbf{E} \left[ \xi + \int_0^T g(s, y_s, z_s) ds \right] + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T.$$

Let  $Y_t := \mathbf{E} \left[ \xi + \int_t^T g(s, y_s, z_s) ds \middle| \mathcal{F}_t \right]$ ,  $0 \leq t \leq T$ . Obviously,  $Y_t \in \mathcal{S}^p(0, T; \mathbf{R}^k)$ . It is not difficult to verify that the  $(Y_t, Z_t)_{t \in [0, T]}$  is just the unique solution in  $L^p$  to the following equation:

$$Y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T]. \quad (3.1)$$

Thus, we have constructed a mapping from  $\mathcal{S}^p(0, T; \mathbf{R}^k) \times M^p(0, T; \mathbf{R}^{k \times d})$  to itself. Denote this mapping by  $\Phi : (y, z) \rightarrow (Y, Z)$ .

In the sequel, suppose that  $(y_t^i, z_t^i)_{t \in [0, T]} \in \mathcal{S}^p(0, T; \mathbf{R}^k) \times M^p(0, T; \mathbf{R}^{k \times d})$ , let  $(Y_t^i, Z_t^i)_{t \in [0, T]}$  be the mapping of  $(y_t^i, z_t^i)_{t \in [0, T]}$ , ( $i = 1, 2$ ), that is  $\Phi(y^i, z^i) = (Y_t^i, Z_t^i)$ ,  $i = 1, 2$ . We denote  $\hat{Y}_t := Y_t^1 - Y_t^2$ ,  $\hat{Z}_t := Z_t^1 - Z_t^2$ ,  $\hat{y}_t := y_t^1 - y_t^2$ ,  $\hat{z}_t := z_t^1 - z_t^2$ ,  $\hat{g}_t := g(t, y_t^1, z_t^1) - g(t, y_t^2, z_t^2)$ . It follows from (3.1) that

$$\hat{Y}_t = \int_t^T \hat{g}_s ds - \int_t^T \hat{Z}_s dB_s, \quad t \in [0, T]. \quad (3.2)$$

Assumption (H3) yields that  $|\hat{g}_t| \leq u(t)|\hat{y}_t| + v(t)|\hat{z}_t|$ , which means that the generator  $\hat{g}_t$  of BSDE (3.2) satisfies assumption (A) with  $\mu(t) = u^{\frac{p-1}{p}}(t)$ ,  $\nu(t) \equiv 0$ ,  $\psi(t, u) \equiv 0$ ,  $\varphi_t = u^{\frac{1}{p}}(t)|\hat{y}_t|$  and  $f_t = v(t)|\hat{z}_t|$  due to the fact that

$$\begin{cases} \mathbf{E} \left[ \int_t^T \left( u^{\frac{1}{p}}(s) |\hat{y}_s| \right)^p ds \right] \leq \hat{u}([t, T]) \cdot \mathbf{E} \left[ \sup_{s \in [t, T]} |\hat{y}_s|^p \right] < +\infty; \\ \mathbf{E} \left[ \left( \int_t^T v(s) |\hat{z}_s| ds \right)^p \right] \leq \hat{v}^{\frac{p}{2}}([t, T]) \cdot \mathbf{E} \left[ \left( \int_t^T |\hat{z}_s|^2 ds \right)^{p/2} \right] < +\infty \end{cases} \quad (3.3)$$

is true for each  $t \in [0, T]$  by Hölder's inequality. Thus, applying Propositions 2.5–2.6 to BSDE (3.2) implies that, in view of (3.3), there exists a constant  $m'_p > 0$

depending only on  $p$  such that for each  $t \in [0, T]$ ,

$$\begin{cases} \mathbf{E} \left[ \left( \int_t^T |\hat{Z}_s|^2 ds \right)^{p/2} \right] \leq m'_p C'([t, T]) \left\{ \hat{v}^{p/2}([t, T]) \cdot \mathbf{E} \left[ \left( \int_t^T |\hat{z}_s|^2 ds \right)^{p/2} \right] \right. \\ \quad \left. + \hat{u}([t, T]) \cdot \mathbf{E} \left[ \sup_{s \in [t, T]} |\hat{y}_s|^p \right] + \mathbf{E} \left[ \sup_{s \in [t, T]} |\hat{Y}_s|^p \right] \right\}, \\ \mathbf{E} \left[ \sup_{s \in [t, T]} |\hat{Y}_s|^p \right] \leq K'([t, T]) \left\{ \frac{1}{2} \hat{u}([t, T]) \cdot \mathbf{E} \left[ \sup_{s \in [t, T]} |\hat{y}_s|^p \right] \right. \\ \quad \left. + m'_p \hat{v}^{p/2}([t, T]) \cdot \mathbf{E} \left[ \left( \int_t^T |\hat{z}_s|^2 ds \right)^{p/2} \right] \right\}. \end{cases}$$

where  $C'([t, T]) := 1 + \hat{u}^{p-1}([t, T]) + \hat{u}^{2p-2}([t, T])$  and  $K'([t, T]) := e^{m'_p \hat{u}([t, T])}$ . Then we have

$$\begin{aligned} \mathbf{E} \left[ \left( \int_t^T |\hat{Z}_s|^2 ds \right)^{p/2} \right] &\leq m'_p C'([t, T]) \left\{ \left( \frac{1}{2} K'([t, T]) + 1 \right) \hat{u}([t, T]) \cdot \mathbf{E} \left[ \sup_{s \in [t, T]} |\hat{y}_s|^p \right] \right. \\ &\quad \left. + (K'([t, T]) m'_p + 1) \hat{v}^{p/2}([t, T]) \cdot \mathbf{E} \left[ \left( \int_t^T |\hat{z}_s|^2 ds \right)^{p/2} \right] \right\}. \end{aligned}$$

Since  $\hat{u}([0, T]) < +\infty$  and  $\hat{v}([0, T]) < +\infty$  by assumption (H3), we can find a positive integer  $N$  and  $0 = T_0 < T_1 < \dots < T_{N-1} < T_N = T$  such that for each  $i = 0, \dots, N-1$ ,

$$\begin{cases} K'([T_i, T_{i+1}]) \frac{\hat{u}([T_i, T_{i+1}])}{2} + m'_p C'([T_i, T_{i+1}]) \left( \frac{1}{2} K'([T_i, T_{i+1}]) + 1 \right) \hat{u}([T_i, T_{i+1}]) \leq \frac{1}{2}; \\ K'([T_i, T_{i+1}]) m'_p \hat{v}^{\frac{p}{2}}([T_i, T_{i+1}]) + m'_p C'([T_i, T_{i+1}]) (K'([T_i, T_{i+1}]) m'_p + 1) \hat{v}^{\frac{p}{2}}([T_i, T_{i+1}]) \leq \frac{1}{2}. \end{cases} \quad (3.4)$$

Based on the above arguments, we can deduce that

$$\begin{aligned} &\mathbf{E} \left[ \sup_{s \in [T_{N-1}, T]} |\hat{Y}_s|^p \right] + \mathbf{E} \left[ \left( \int_{T_{N-1}}^T |\hat{Z}_s|^2 ds \right)^{p/2} \right] \\ &\leq \frac{1}{2} \left\{ \mathbf{E} \left[ \sup_{s \in [T_{N-1}, T]} |\hat{y}_s|^p \right] + \mathbf{E} \left[ \left( \int_{T_{N-1}}^T |\hat{z}_s|^2 ds \right)^{p/2} \right] \right\}, \end{aligned}$$

which means that  $\Phi$  is a strict contraction from  $\mathcal{S}^p(T_{N-1}, T; \mathbf{R}^k) \times M^p(T_{N-1}, T; \mathbf{R}^{k \times d})$  into itself. Then  $\Phi$  has a unique fixed point in this space. It follows that there exists a unique  $(y_t, z_t)_{t \in [T_{N-1}, T]} \in \mathcal{S}^p(T_{N-1}, T; \mathbf{R}^k) \times M^p(T_{N-1}, T; \mathbf{R}^{k \times d})$  satisfying the BSDE with parameters  $(\xi, T, g)$  on  $[T_{N-1}, T]$ . That is to say, the BSDE has a unique solution in  $L^p$  on  $[T_{N-1}, T]$ . Finally, note that (3.4) holds true for  $i = N-2$ . By replacing  $T_{N-1}, T$  and  $\xi$  by  $T_{N-2}, T_{N-1}$  and  $y_{T_{N-1}}$  respectively in the above proof except for the paragraph containing (3.4), we can obtain the existence and uniqueness of a solution in  $L^p$  to the BSDE with parameters  $(\xi, T, g)$  on  $[T_{N-2}, T_{N-1}]$ . Furthermore, repeating the above procedure and making use of (3.4), we deduce the existence and uniqueness of a solution in  $L^p$  to the BSDE with parameters  $(\xi, T, g)$  on  $[T_{N-3}, T_{N-2}], \dots, [0, T_1]$ . The proof of Proposition 3.3 is then completed.  $\square$

Now, we are in a position to prove Theorem 3.2. Let  $0 < T \leq +\infty$ ,  $\xi \in L^p(\mathbf{R}^k)$  and  $g$  satisfy (H4) and (H5). We can construct the Picard approximation sequence of the BSDE with parameters  $(\xi, T, g)$  as follows:

$$y_t^0 = 0; \quad y_t^n = \xi + \int_t^T g(s, y_s^{n-1}, z_s^n) ds - \int_t^T z_s^n dB_s, \quad t \in [0, T]. \quad (3.5)$$

Indeed, for each  $n \geq 1$ , it follows from assumption (H4) that

$$\begin{aligned} |g(s, y_s^{n-1}, 0)| &\leq |g(s, 0, 0)| + \alpha(s) \rho^{\frac{1}{p}}(s, |y_s^{n-1}|^p) \\ &\leq |g(s, 0, 0)| + \alpha(s) \left( a^{\frac{1}{p}}(s) + b^{\frac{1}{p}}(s) |y_s^{n-1}| \right), \end{aligned}$$

and then

$$\begin{aligned} \mathbf{E} \left[ \left( \int_0^T |g(s, y_s^{n-1}, 0)| ds \right)^p \right] &\leq 3^p \mathbf{E} \left[ \left( \int_0^T |g(s, 0, 0)| ds \right)^p \right] + 3^p \left( \int_0^T \alpha(s) a^{\frac{1}{p}}(s) ds \right)^p \\ &\quad + 3^p \left( \int_0^T \alpha(s) b^{\frac{1}{p}}(s) ds \right)^p \mathbf{E} \left[ \sup_{s \in [0, T]} |y_s^{n-1}|^p \right]. \end{aligned}$$

Furthermore, by Remark 3.1 and assumption (H4), the generator  $g(s, y_s^{n-1}, z)$  of BSDE (3.5) satisfies (H5) and (H3) with  $u(t) = 0$  and  $v(t) = \beta(t)$ . It follows from Proposition 3.3 that the equation (3.5) has a unique solution  $(y_t^n, z_t^n)_{t \in [0, T]}$  in  $L^p$  for each  $n \geq 1$ . With respect to the processes  $(y_t^n, z_t^n)_{t \in [0, T]}$ , we have the following Lemma 3.4 and Lemma 3.5. For notational convenience, in the following for each  $0 \leq t_1 \leq t_2 \leq T$ , we define

$$\hat{\alpha}([t_1, t_2]) := \int_{t_1}^{t_2} \alpha^{\frac{p}{p-1}}(s) ds \quad \text{and} \quad \hat{\beta}([t_1, t_2]) := \int_{t_1}^{t_2} \beta^2(s) ds.$$

**Lemma 3.4:** *Under the hypotheses of Theorem 3.2, there exists a constant  $\bar{m}_p > 0$  depending only on  $p$  such that for each  $t \in [0, T]$ ,  $n, m \geq 1$ ,*

$$\begin{cases} \mathbf{E} \left[ \left( \int_t^T |z_s^{n+m} - z_s^n|^2 ds \right)^{p/2} \right] \leq \bar{m}_p \bar{C}([t, T]) \left\{ \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s^{n+m} - y_s^n|^p \right] \right. \\ \quad \left. + \int_t^T \rho(s, \mathbf{E} [|y_s^{n+m-1} - y_s^{n-1}|^p]) ds \right\}; \\ \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s^{n+m} - y_s^n|^p \right] \leq \frac{1}{2} \bar{K}([t, T]) \int_t^T \rho(s, \mathbf{E} [|y_s^{n+m-1} - y_s^{n-1}|^p]) ds, \end{cases} \quad (3.6)$$

where

$$\begin{cases} \bar{C}([t, T]) := 1 + \hat{\alpha}^{p-1}([t, T]) + \hat{\alpha}^{2p-2}([t, T]) + \hat{\beta}^{p/2}([t, T]) + \hat{\beta}^p([t, T]); \\ \bar{K}([t, T]) := e^{\bar{m}_p(\hat{\alpha}([t, T]) + \hat{\beta}([t, T]))}. \end{cases}$$

**Proof:** It follows from (3.5) that the process  $(y_t^{n+m} - y_t^n, z_t^{n+m} - z_t^n)_{t \in [0, T]}$  is a solution of the following BSDE:

$$y_t = \int_t^T f_{n,m}(s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T], \quad (3.7)$$

where  $f_{n,m}(s, z) := g(s, y_s^{n+m-1}, z + z_s^n) - g(s, y_s^{n-1}, z_s^n)$ . It follows from assumption (H3) that  $|f_{n,m}(s, z)| \leq \alpha(s)\rho^{\frac{1}{p}}(s, |y_s^{n+m-1} - y_s^{n-1}|^p) + \beta(s)|z|$ , which means that assumption (A) is satisfied for the generator  $f_{n,m}(t, z)$  of BSDE (3.7) with  $\mu(t) = \alpha(t)$ ,  $\nu(t) = \beta(t)$ ,  $\psi(t, u) \equiv 0$ ,  $f_t \equiv 0$  and  $\varphi_t = \rho^{\frac{1}{p}}(t, |y_t^{n+m-1} - y_t^{n-1}|^p)$  by Remark 2.4. Thus, in view of the fact that  $\rho(s, \cdot)$  is a concave function for each  $s \in [0, T]$ , the conclusion (3.6) follows from Proposition 2.5, Proposition 2.6, and then Fubini's theorem and Jensen's inequality. Lemma 3.4 is proved.  $\square$

**Lemma 3.5:** *Under the hypotheses of Theorem 3.2, there exists a constant  $\hat{m}_p > 0$  depending only on  $p$  such that for each  $n \geq 1$  and each  $t \in [0, T]$ ,*

$$\mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^n|^p \right] \leq \hat{C}([t, T]) + \frac{1}{2} e^{\hat{m}_p(\hat{\alpha}([t, T]) + \hat{\beta}([t, T]))} \int_t^T \rho(s, \mathbf{E}[|y_s^{n-1}|^p]) ds, \quad (3.8)$$

where

$$\hat{C}([t, T]) := \hat{m}_p e^{\hat{m}_p(\hat{\alpha}([t, T]) + \hat{\beta}([t, T]))} \left\{ \mathbf{E}|\xi|^p + \mathbf{E} \left[ \left( \int_t^T |g(s, 0, 0)| ds \right)^p \right] \right\}.$$

**Proof:** It follows from the hypotheses of Theorem 3.2 that

$$\begin{aligned} |g(s, y_s^{n-1}, z)| &\leq |g(s, y_s^{n-1}, z) - g(s, 0, 0)| + |g(s, 0, 0)| \\ &\leq \alpha(s)\rho^{\frac{1}{p}}(s, |y_s^{n-1}|^p) + \beta(s)|z| + |g(s, 0, 0)|. \end{aligned}$$

Then, assumption (A) is satisfied for the generator  $g(s, y_s^{n-1}, z)$  of BSDE (3.5) with  $\mu(t) = \alpha(t)$ ,  $\nu(t) = \beta(t)$ ,  $\psi(t, u) \equiv 0$ ,  $f_t = |g(t, 0, 0)|$  and  $\varphi_t = \rho^{\frac{1}{p}}(t, |y_t^{n-1}|^p)$  by Remark 2.4. Thus, in view of the fact that  $\rho(t, \cdot)$  is a concave function for each  $t \in [0, T]$ , (3.8) follows from Proposition 2.6 and then Fubini's Theorem and Jensen's inequality. Lemma 3.5 is proved.  $\square$

In the sequel, since  $\hat{\alpha}([0, T]) < +\infty$  and  $\hat{\beta}([0, T]) < +\infty$  by assumption (H4), we can find a positive integer  $\bar{N}$  and  $0 = \bar{T}_0 < \bar{T}_1 < \dots < \bar{T}_{\bar{N}-1} < \bar{T}_{\bar{N}} = T$  such that for each  $i = 0, \dots, \bar{N} - 1$ ,

$$\int_{\bar{T}_i}^{\bar{T}_{i+1}} b(s) ds \leq \frac{1}{2} \quad \text{and} \quad \hat{\alpha}([\bar{T}_i, \bar{T}_{i+1}]) + \hat{\beta}([\bar{T}_i, \bar{T}_{i+1}]) \leq \frac{\ln 2}{\hat{m}_p} \wedge \frac{\ln 2}{\bar{m}_p}, \quad (3.9)$$

where  $\bar{m}_p$  and  $\hat{m}_p$  are respectively defined in Lemma 3.4 and Lemma 3.5.

With the help of Lemma 3.4 and Lemma 3.5, we can prove Theorem 3.2.

**Proof of Theorem 3.2.** Existence: Let us set  $M = 2\hat{C}([0, T]) + 2 \int_0^T a(s) ds \geq 0$ . It follows from (H4) and (3.9) that for each  $t \in [\bar{T}_{\bar{N}-1}, T]$ ,

$$\hat{C}([0, T]) + \int_t^T \rho(s, M) ds \leq \hat{C}([0, T]) + \int_t^T a(s) ds + M \int_t^T b(s) ds \leq \frac{M}{2} + \frac{M}{2} = M, \quad (3.10)$$

and from Lemma 3.5 and (3.9) that

$$\mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^n|^p \right] \leq \hat{C}([0, T]) + \int_t^T \rho(s, \mathbf{E}[|y_s^{n-1}|^p]) ds, \quad t \in [\bar{T}_{\bar{N}-1}, T]. \quad (3.11)$$

Since  $\rho(s, \cdot)$  is a nondecreasing function for each  $s \in [0, T]$ , by (3.11) and (3.10) we can deduce that for each  $t \in [\bar{T}_{\bar{N}-1}, T]$ ,  $\mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^1|^p \right] \leq \hat{C}([0, T]) \leq M$ ,

$$\begin{aligned} \mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^2|^p \right] &\leq \hat{C}([0, T]) + \int_t^T \rho(s, \mathbf{E}[|y_s^1|^p]) ds \leq \hat{C}([0, T]) + \int_t^T \rho(s, M) ds \leq M, \\ \mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^3|^p \right] &\leq \hat{C}([0, T]) + \int_t^T \rho(s, \mathbf{E}[|y_s^2|^p]) ds \leq \hat{C}([0, T]) + \int_t^T \rho(s, M) ds \leq M. \end{aligned}$$

Thus, by induction we know that for each  $n \geq 1$  and each  $t \in [\bar{T}_{\bar{N}-1}, T]$ ,

$$\mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^n|^p \right] \leq M. \quad (3.12)$$

Now, we define a sequence of functions  $\{\varphi_n(t)\}_{n \geq 1}$  as follows:

$$\varphi_0(t) = \int_t^T \rho(s, M) ds; \quad \varphi_{n+1}(t) = \int_t^T \rho(s, \varphi_n(s)) ds. \quad (3.13)$$

For all  $t \in [\bar{T}_{\bar{N}-1}, T]$ , it follows from (3.10) that  $\varphi_0(t) = \int_t^T \rho(s, M) ds \leq M$ . Furthermore, by induction we can obtain that for all  $n \geq 1$ ,  $\varphi_n(t)$  satisfies  $0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq \dots \leq \varphi_1(t) \leq \varphi_0(t) \leq M$ . Then, for each  $t \in [\bar{T}_{\bar{N}-1}, T]$ , the limit of the sequence  $\{\varphi_n(t)\}_{n \geq 1}$  must exist: we denote it by  $\varphi(t)$ . Letting  $n \rightarrow \infty$  in (3.13), in view of the fact that  $\rho(s, \cdot)$  is a continuous function for each  $s \in [0, T]$ ,  $\rho(s, \varphi_n(s)) \leq \rho(s, M)$  for each  $n \geq 1$ , and  $\int_t^T \rho(s, M) ds \leq M$ , we can deduce from Lebesgue's dominated convergence theorem that for each  $t \in [\bar{T}_{\bar{N}-1}, T]$ ,  $\varphi(t) = \int_t^T \rho(s, \varphi(s)) ds$ , whether  $T < +\infty$  or  $T = +\infty$ . Then, by virtue of (H4) we know that  $\varphi(t) = 0$ ,  $t \in [\bar{T}_{\bar{N}-1}, T]$ .

In the sequel, for each  $t \in [\bar{T}_{\bar{N}-1}, T]$  and  $n, m \geq 1$ , it follows from Lemma 3.4, (3.9) and (3.12) that, whether  $T < +\infty$  or  $T = +\infty$ ,

$$\begin{aligned} \mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^{1+m} - y_r^1|^p \right] &\leq \int_t^T \rho(s, \mathbf{E}[|y_s^m|^p]) ds \leq \int_t^T \rho(s, M) ds = \varphi_0(t) \leq M, \\ \mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^{2+m} - y_r^2|^p \right] &\leq \int_t^T \rho(s, \mathbf{E}[|y_s^{1+m} - y_s^1|^p]) ds \leq \int_t^T \rho(s, \varphi_0(s)) ds = \varphi_1(t), \\ \mathbf{E} \left[ \sup_{r \in [t, T]} |y_r^{3+m} - y_r^3|^p \right] &\leq \int_t^T \rho(s, \mathbf{E}[|y_s^{2+m} - y_s^2|^p]) ds \leq \int_t^T \rho(s, \varphi_1(s)) ds = \varphi_2(t). \end{aligned}$$

Thus, by induction we can derive that for each  $m \geq 1$ ,

$$\mathbf{E} \left[ \sup_{\bar{T}_{\bar{N}-1} \leq r \leq T} |y_r^{n+m} - y_r^n|^p \right] \leq \varphi_{n-1}(\bar{T}_{\bar{N}-1}) \rightarrow 0, \quad n \rightarrow \infty.$$

which means that  $\{y_t^n\}_{n \geq 1}$  is a Cauchy sequence in  $\mathcal{S}^p(\bar{T}_{\bar{N}-1}, T; \mathbf{R}^k)$ . Furthermore, since  $\rho(s, \cdot)$  is continuous and  $\rho(s, 0) = 0$  for each  $s \in [0, T]$ ,  $\int_{\bar{T}_{\bar{N}-1}}^T \rho(s, M) ds \leq M$ , and  $\rho(s, \mathbf{E}[|y_s^{n+m-1} - y_s^{n-1}|^p]) \leq \rho(s, M)$  for each  $s \in [\bar{T}_{\bar{N}-1}, T]$ , we also know from (3.6) and Lebesgue's dominated convergence theorem that  $\{z_t^n\}_{n \geq 1}$  is a Cauchy sequence in  $M^p(\bar{T}_{\bar{N}-1}, T; \mathbf{R}^{k \times d})$ . Define their limits by  $(y_t)_{t \in [\bar{T}_{\bar{N}-1}, T]}$  and

$(z_t)_{t \in [\bar{T}_{\bar{N}-1}, T]}$  respectively. Letting  $n \rightarrow \infty$  in (3.5) implies that  $(y_t, z_t)$  is a solution in  $L^p$  to the BSDE with parameters  $(\xi, T, g)$  on  $[\bar{T}_{\bar{N}-1}, T]$ .

Finally, note that (3.9) holds true for  $i = \bar{N} - 2$ . By replacing  $\bar{T}_{\bar{N}-1}$ ,  $T$  and  $\xi$  with  $\bar{T}_{\bar{N}-2}$ ,  $\bar{T}_{\bar{N}-1}$  and  $y_{\bar{T}_{\bar{N}-1}}$  respectively in the above arguments beginning from the end of the proof of Proposition 3.3 (except for the paragraph containing (3.9)), we can obtain the existence of a solution in  $L^p$  to the BSDE with parameters  $(\xi, T, g)$  on  $[\bar{T}_{\bar{N}-2}, \bar{T}_{\bar{N}-1}]$ . Furthermore, repeating the above procedure and making use of (3.9), we deduce the existence of a solution in  $L^p$  to the BSDE with parameters  $(\xi, T, g)$  on  $[\bar{T}_{\bar{N}-3}, \bar{T}_{\bar{N}-2}]$ ,  $\dots$ ,  $[0, \bar{T}_1]$ . This proves the existence.

Uniqueness: Let  $(y_t^i, z_t^i)_{t \in [0, T]}$  ( $i = 1, 2$ ) be two solutions in  $L^p$  of the BSDE with parameters  $(\xi, T, g)$ . Then,  $(y_t^1 - y_t^2, z_t^1 - z_t^2)_{t \in [0, T]}$  is a solution in  $L^p$  to the following BSDE:

$$y_t = \int_t^T \hat{g}(s, y_s, z_s) \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T], \quad (3.14)$$

where  $\hat{g}(s, y, z) := g(s, y + y_s^2, z + z_s^2) - g(s, y_s^2, z_s^2)$ . It follows from (H4) that  $|\hat{g}(s, y, z)| \leq \alpha(s)\rho^{\frac{1}{p}}(s, |y|^p) + \beta(s)|z|$ , which means that assumption (A) is satisfied for the generator  $\hat{g}(t, y, z)$  of BSDE (3.14) with  $\mu(t) = \alpha(t)$ ,  $\nu(t) = \beta(t)$ ,  $\psi(t, u) = \rho(t, u)$ ,  $\varphi_t \equiv 0$  and  $f_t \equiv 0$ . Then, Proposition 2.5 and Proposition 2.6 yield that there exists a constant  $\tilde{m}_p > 0$  depending only on  $p$  such that for  $t \in [0, T]$ ,

$$\begin{cases} \mathbf{E} \left[ \left( \int_t^T |z_s^1 - z_s^2|^2 \, ds \right)^{p/2} \right] \leq \tilde{m}_p \tilde{C}([t, T]) \left\{ \mathbf{E} \left[ \sup_{s \in [t, T]} |y_s^1 - y_s^2|^p \right] + \int_t^T \rho(s, \mathbf{E}[|y_s^1 - y_s^2|^p]) \, ds \right\}; \\ \mathbf{E} [|y_t^1 - y_t^2|^p] \leq \frac{1}{2} e^{\tilde{m}_p(\hat{\alpha}([t, T]) + \hat{\beta}([t, T]))} \int_t^T \rho(s, \mathbf{E}[|y_s^1 - y_s^2|^p]) \, ds, \end{cases} \quad (3.15)$$

where  $\tilde{C}([t, T]) := 1 + \hat{\alpha}^{p-1}([t, T]) + \hat{\alpha}^{2p-2}([t, T]) + \hat{\beta}^{p/2}([t, T]) + \hat{\beta}^p([t, T])$ .

Similar to (3.9), we can find a positive integer  $\tilde{N}$  and  $0 = \tilde{T}_0 < \tilde{T}_1 < \dots < \tilde{T}_{\tilde{N}-1} < \tilde{T}_{\tilde{N}} = T$  such that for each  $i = 0, \dots, \tilde{N} - 1$ ,

$$\hat{\alpha}([\tilde{T}_i, \tilde{T}_{i+1}]) + \hat{\beta}([\tilde{T}_i, \tilde{T}_{i+1}]) \leq \frac{\ln 2}{\tilde{m}_p}. \quad (3.16)$$

Then, it follows from (3.15) and (3.16) that for each  $t \in [\tilde{T}_{\tilde{N}-1}, T]$ ,

$$\mathbf{E} [|y_t^1 - y_t^2|^p] \leq \int_t^T \rho(s, \mathbf{E}[|y_s^1 - y_s^2|^p]) \, ds. \quad (3.17)$$

From the ODE comparison theorem, we know that  $\mathbf{E}[|y_t^1 - y_t^2|^p] \leq r(t)$ , where  $r(t)$  is the maximum left shift solution of the following equation:

$$u'(t) = -\rho(t, u); \quad u(T) = 0.$$

It follows from (H4) that  $r(t) = 0$ ,  $t \in [\tilde{T}_{\tilde{N}-1}, T]$ . Hence,  $\mathbf{E}[|y_t^1 - y_t^2|^p] = 0$ ,  $t \in [\tilde{T}_{\tilde{N}-1}, T]$ , which means  $y_t^1 = y_t^2$  for each  $t \in [\tilde{T}_{\tilde{N}-1}, T]$ . Furthermore, (3.15) implies

that  $z_t^1 = z_t^2$  holds true almost surely for each  $t \in [\tilde{T}_{\tilde{N}-1}, T]$ . Thus, we have obtained the uniqueness result on  $[\tilde{T}_{\tilde{N}-1}, T]$ . Then, thanks to (3.16), we can repeat the above for the proof of uniqueness by replacing  $\tilde{T}_{\tilde{N}-1}$ ,  $T$  and  $\xi$  with  $\tilde{T}_{\tilde{N}-2}$ ,  $\tilde{T}_{\tilde{N}-1}$  and  $y_{\tilde{T}_{\tilde{N}-1}}$  respectively to obtain the uniqueness result on  $[\tilde{T}_{\tilde{N}-2}, \tilde{T}_{\tilde{N}-1}]$  and then on the whole  $[0, T]$ . The proof of Theorem 3.2 is complete.  $\square$

#### 4. Examples, Corollaries and Remarks

In this section, we will introduce some examples, corollaries and remarks to show that Theorem 3.2 of this paper is a generalization of the main results in Pardoux and Peng [11], Mao [9], Chen [4], Chen and Wang [5], Wang and Wang [13] and Wang and Huang [12]. Firstly, by Remark 3.1 and Theorem 3.2, the following corollary is immediate, which generalizes the main results in Mao [9], Wang and Wang [13] and Wang and Huang [12].

**Corollary 4.1:** *Let  $0 < T < +\infty$  and  $g$  satisfy (H4) and  $g(\cdot, 0, 0) \in M^p(0, T; \mathbf{R}^k)$ . Then, for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .*

**Remark 4.2:** Theorem 3.2 in Wang and Huang [12] proved that if  $0 < T < +\infty$ ,  $g$  satisfies assumption (H2), and  $g(\cdot, 0, 0) \in M^2(0, T; \mathbf{R}^k)$ , then the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^2$ . This result can be regarded as an immediate consequence of Corollary 4.1. Indeed, it follows that if  $g$  satisfies (H2), then  $g$  must satisfy (H4) with  $p = 2$ ,  $\alpha(t) \equiv 1$ ,  $\beta(t) \equiv \sqrt{c}$  and  $\rho(t, u) = \kappa(t, u)$ .

Furthermore, let us introduce the following assumption, where we assume that  $0 < T \leq +\infty$ :

$$(H6) \quad dP \times dt - a.s., \quad \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d},$$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq b(t)\bar{\kappa}^{\frac{1}{p}}(|y_1 - y_2|^p) + c(t)|z_1 - z_2|,$$

where  $b(\cdot), c(\cdot) : [0, T] \mapsto \mathbf{R}^+$  satisfy  $\int_0^T [b(t) + c^2(t)] dt < +\infty$  and  $\bar{\kappa}(\cdot)$  is a concave and nondecreasing function from  $\mathbf{R}^+$  to  $\mathbf{R}^+$  such that  $\bar{\kappa}(0) = 0$ ,  $\bar{\kappa}(u) > 0$  for  $u > 0$ , and  $\int_{0+} \bar{\kappa}^{-1}(u) du = +\infty$ .

**Remark 4.3:** In next section, we will show that the concavity condition of  $\bar{\kappa}(\cdot)$  in (H6) can be weakened to the continuity condition and that the bigger the  $p$ , the stronger the (H6).

The assumptions of  $\bar{\kappa}(\cdot)$  in (H6) yield that there exists a constant  $A > 0$  such that for each  $u \geq 0$ ,  $\bar{\kappa}(u) \leq Au + A$ . Then, from Theorem 3.2 and Bihari's inequality, letting  $\alpha(t) = b^{\frac{p-1}{p}}(t)$ ,  $\beta(t) = c(t)$  and  $\rho(t, u) = b(t)\bar{\kappa}(u) \in \mathbf{S}[T, Ab(t), Ab(t)]$  in (H4), we can obtain the following corollary.

**Corollary 4.4:** *Let  $0 < T \leq +\infty$  and  $g$  satisfy (H5) and (H6). Then, for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .*

**Remark 4.5:** Theorem 2.1 in Mao [9] proved that if  $0 < T < +\infty$ ,  $g(\cdot, 0, 0) \in M^2(0, T; \mathbf{R}^k)$  and  $g$  satisfies assumption (H1), then the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^2$ . This result can be regarded as an immediate consequence of Corollary 4.4. Indeed, it follows from Remark 3.1 that under the above assumptions, the generator  $g$  must satisfy (H5) and (H6) with  $p = 2$ ,  $b(t) \equiv 1$ ,  $c(t) \equiv \sqrt{c}$  and  $\bar{\kappa}(u) = \kappa(u)$ .



**Example 4.6** Let  $0 < T < +\infty$ , and let

$$g(t, y, z) = \frac{1}{\sqrt{t}} h(|y|) + \frac{1}{\sqrt[4]{t}} |z| + |B_t|,$$

where  $h(x) := x |\ln x|^{1/p} \cdot 1_{0 < x \leq \delta} + (h'(\delta-)(x - \delta) + h(\delta)) \cdot 1_{x > \delta}$  with  $\delta > 0$  small enough. It is not difficult to verify that  $g$  satisfies assumptions (H5) and (H6) with  $b(t) = 1/\sqrt{t}$ ,  $c(t) = 1/\sqrt[4]{t}$  and  $\bar{\kappa}(u) = h^p(u^{1/p})$ . Thus, Corollary 4.4 yields that for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .

**Remark 4.7:** Proposition 3.3 in the beginning of Section 3 can also be regarded as an immediate consequence of Corollary 4.4. Indeed, if  $g$  satisfies (H3), then  $g$  satisfies (H6) with  $b(t) = u(t)$ ,  $c(t) = v(t)$  and  $\bar{\kappa}(u) = u$  for  $u \geq 0$ .

In the following, we introduce an example where  $T$  can be  $+\infty$ .

**Example 4.8** Let  $0 < T \leq +\infty$ , and let

$$g(t, y, z) = \frac{1}{(1+t)^2} \sigma(|y|) + \frac{1}{1+t} |z| + \frac{1}{(1+t)^2},$$

where  $\sigma(x) := x(|\ln x| \ln |\ln x|)^{1/p} \cdot 1_{0 < x \leq \delta} + (\sigma'(\delta-)(x - \delta) + \sigma(\delta)) \cdot 1_{x > \delta}$  with  $\delta > 0$  small enough. It is not difficult to verify that  $g$  satisfies assumptions (H5) and (H6) with  $b(t) = 1/(1+t)^2$ ,  $c(t) = 1/(1+t)$  and  $\bar{\kappa}(u) = \sigma^p(u^{1/p})$ . Thus, it follows from Corollary 4.4 that for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .

Finally, let us make Remark 4.9, which illustrates an important difference between the infinite and finite  $T$  cases.

**Remark 4.9:** It is clear that in the case where  $T < +\infty$ , the  $(y_t)_{t \in [0, T]}$  among the solution  $(y_t, z_t)_{t \in [0, T]}$  of BSDEs discussed in this paper belongs also to  $\mathbf{MP}(0, T; \mathbf{R}^k)$ . However, in the case where  $T = +\infty$ , this conclusion does not hold true. For a simple example, letting  $\xi \equiv 1$ ,  $T = +\infty$  and  $g \equiv 0$ , from Theorem 3.2 or Corollary 4.4 we know that the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ . Obviously, this solution is just  $(1, 0)_{0 \leq t \leq +\infty}$ . The process  $(1)_{0 \leq t \leq +\infty}$  belongs to  $S^p(0, T; \mathbf{R}^k)$ , but it does not belong to  $\mathbf{MP}(0, T; \mathbf{R}^k)$ .

## 5. Further Discussion

In this section, some further discussions with respect to our main result will be given. First, let us examine Remark 4.3. We need to show that the concavity condition of  $\bar{\kappa}(\cdot)$  in (H6) can be weakened to the continuity condition and that the bigger the  $p$ , the stronger the (H6). To be precise, we need to prove that if  $g$  satisfies the following assumption (H7') with  $q \geq p$ , then  $g$  must satisfy the following assumption (H7).

(H7) There exists a deterministic function  $b(t) : [0, T] \rightarrow \mathbf{R}^+$  with  $\int_0^T b(t) dt < +\infty$  and a nondecreasing and concave function  $\kappa(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$  with  $\kappa(0) = 0$ ,  $\kappa(u) > 0$  for  $u > 0$ , and  $\int_{0+} \kappa^{-1}(u) du = +\infty$  such that  $dP \times dt - a.s.$ ,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq b(t) \kappa^{\frac{1}{p}}(|y_1 - y_2|^p).$$

(H7') There exists a deterministic function  $\bar{b}(t) : [0, T] \rightarrow \mathbf{R}^+$  with  $\int_0^T \bar{b}(t) dt < +\infty$  and a nondecreasing and continuous function  $\bar{\kappa}(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$  with  $\bar{\kappa}(0) = 0$ ,



$\bar{\kappa}(u) > 0$  for  $u > 0$ , and  $\int_{0+} \bar{\kappa}^{-1}(u) du = +\infty$  such that  $dP \times dt - a.s.$ ,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \bar{b}(t) \bar{\kappa}^{\frac{1}{q}}(|y_1 - y_2|^q).$$

In order to show this fact, we need the following technical Lemma proved in the Appendix.

**Lemma 5.1:** *Let  $\rho(\cdot)$  be a nondecreasing and concave function on  $\mathbf{R}^+$  with  $\rho(0) = 0$ . Then we have*

$$\forall r > 1, \quad \rho^r(x^{1/r}) \text{ is also a nondecreasing and concave function on } \mathbf{R}^+. \quad (5.1)$$

Moreover, if  $\rho(u) > 0$  for  $u > 0$ , and  $\int_{0+} \rho^{-1}(u) du = +\infty$ , then

$$\forall r < 1, \quad \int_{0+} \frac{du}{\rho^r(u^{1/r})} = +\infty. \quad (5.2)$$

Now, we can show that  $(H7') \implies (H7)$ . Let us assume that  $(H7')$  holds true for  $g$ . Then we have,  $dP \times dt - a.s.$ ,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \bar{b}(t) \rho_1(|y_1 - y_2|),$$

where  $\rho_1(u) := \bar{\kappa}^{\frac{1}{q}}(u^q)$ . Obviously,  $\rho_1(u)$  is a continuous and nondecreasing function on  $\mathbf{R}_+$  with  $\rho_1(0) = 0$  and  $\rho_1(u) > 0$  for  $u > 0$ , but it is not necessary to be concave. However, it follows from the classical theory of uniformly continuous functions that if  $g$  satisfies the above inequality, then there exists a concave and nondecreasing function  $\rho_2(\cdot)$  such that  $\rho_2(0) = 0$ ,  $\rho_2(u) \leq 2\rho_1(u)$  for  $u \geq 0$ , and  $dP \times dt - a.s.$ ,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \bar{b}(t) \rho_2(|y_1 - y_2|).$$

Thus,  $dP \times dt - a.s.$ ,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \bar{b}(t) \kappa^{\frac{1}{p}}(|y_1 - y_2|^p),$$

where  $\kappa(u) := \rho_2^p(u^{\frac{1}{p}}) + u$ . It is clear that  $\kappa(0) = 0$  and  $\kappa(u) > 0$  for  $u > 0$ . Moreover, it follows from (5.1) in Lemma 5.1 that  $\kappa(\cdot)$  is also a nondecreasing and concave function due to the fact that  $p > 1$  and  $\rho_2(\cdot)$  is a nondecreasing and concave function. Thus, to prove that  $(H7)$  holds, it suffices to show that  $\int_{0+} \kappa^{-1}(u) du = +\infty$ . Indeed, if  $\rho_2(1) = 0$ , then since  $\rho_2(u) = 0$  for each  $u \in [0, 1]$ , we have  $\int_{0+} \kappa^{-1}(u) du = \int_{0+} u^{-1} du = +\infty$ . On the other hand, if  $\rho_2(1) > 0$ , since  $\rho_2(\cdot)$  is a concave function with  $\rho_2(0) = 0$ , we know that

$$\rho_2(u) = \rho_2(u \cdot 1 + (1 - u) \cdot 0) \geq u\rho_2(1) + (1 - u)\rho_2(0) = u\rho_2(1), \quad u \in [0, 1], \quad (5.3)$$

and then  $\rho_2^p(u^{\frac{1}{p}}) \geq \left(u^{\frac{1}{p}} \rho_2(1)\right)^p = \rho_2^p(1)u$ . Thus, we have

$$\forall u \geq 0, \quad \kappa(u) = \rho_2^p(u^{\frac{1}{p}}) + u \leq K\rho_2^p(u^{\frac{1}{p}}) \leq K2^p \rho_1^p(u^{\frac{1}{p}}) = K2^p \bar{\kappa}^{\frac{p}{q}}(u^{\frac{a}{p}}),$$

where  $K = 1 + 1/\rho_2^p(1)$ . Consequently, if  $q = p$ , then

$$\int_{0+} \frac{du}{\kappa(u)} \geq \frac{1}{K2^p} \int_{0+} \frac{du}{\bar{\kappa}(u)} = +\infty.$$

Thus, we have proved that (H7') with  $q = p$  implies (H7). As a result, we can now assume that the  $\bar{\kappa}(\cdot)$  in (H7') is a concave function. Then, if  $q > p$ , from (5.2) of Lemma 5.1 with  $\rho(\cdot) = \bar{\kappa}(\cdot)$  and  $r = p/q < 1$  we have

$$\int_{0+} \frac{du}{\bar{\kappa}(u)} \geq \frac{1}{K2^p} \int_{0+} \frac{du}{\bar{\kappa}^{\frac{p}{q}}(u^{\frac{q}{p}})} = +\infty.$$

Hence (H7')  $\implies$  (H7), i.e., the concavity condition of  $\bar{\kappa}(\cdot)$  in (H6) can be weakened to the continuity condition and the bigger the  $p$ , the stronger the (H6).

Furthermore, let us introduce the following assumption (H6\*), where we also assume that  $0 < T \leq +\infty$ :

$$(H6^*) \quad dP \times dt - a.s., \quad \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d},$$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq b(t)\kappa(|y_1 - y_2|) + c(t)|z_1 - z_2|,$$

where  $b(\cdot), c(\cdot) : [0, T] \mapsto \mathbf{R}^+$  satisfy  $\int_0^T [b(t) + c^2(t)] dt < +\infty$  and  $\kappa(\cdot)$  is a continuous and nondecreasing function from  $[0, T]$  to  $\mathbf{R}^+$  such that  $\kappa(0) = 0$ ,  $\kappa(u) > 0$  for  $u > 0$ , and  $\int_{0+} \kappa^{-p}(u)u^{p-1} du = +\infty$ .

In the following, we will show (H6\*)  $\implies$  (H6). In fact, if  $g$  satisfies (H6\*), then we have,  $dP \times dt - a.s., \quad \forall y_1, y_2 \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d}$ ,

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq b(t)\bar{\kappa}^{\frac{1}{p}}(|y_1 - y_2|^p) + c(t)|z_1 - z_2|,$$

where  $\bar{\kappa}(u) = \kappa^p(u^{1/p})$ . And, we have also

$$\int_{0+} \frac{du}{\bar{\kappa}(u)} = \int_{0+} \frac{du}{\kappa^p(u^{\frac{1}{p}})} = \int_{0+} \frac{pu^{p-1}}{\kappa^p(u)} du = +\infty. \quad (5.4)$$

Thus, it follows from Remark 4.3 that (H6) is true. Therefore, from Corollary 4.4 the following corollary is immediate. It follows from Hölder's inequality that it generalizes the corresponding result in Constantin [6] where  $p = 2$ , (H5) is replaced with  $g(\cdot, 0, 0) \in M^2(0, T; \mathbf{R}^k)$ , and  $b(t) \equiv 1$ ,  $c(t) \equiv c$  in (H6\*).

**Corollary 5.2:** *Let  $0 < T \leq +\infty$  and  $g$  satisfy (H5) and (H6\*). Then, for each  $\xi \in L^p(\mathbf{R}^k)$ , the BSDE with parameters  $(\xi, T, g)$  has a unique solution in  $L^p$ .*

**Remark 5.3:** According to the classical theory of uniformly continuous functions, we can assume that the  $\kappa(\cdot)$  in (H6\*) is a concave function. Thus, applying (5.2) of Lemma 5.1 yields, by letting  $\rho(u) = \kappa^q(u^{1/q})$  and  $r = p/q$  with  $q > p$ , that if  $\int_{0+} \kappa^{-q}(u^{1/q}) du = +\infty$ , then  $\int_{0+} \kappa^{-p}(u^{1/p}) du = +\infty$ . As a result, noticing (5.4) we know that the bigger the  $p$ , the stronger the (H6\*).

Finally, it should be noted that the conclusions of Example 4.6 and Example 4.8 can also be obtained by virtue of Corollary 5.2.

## 6. Appendix

**Proof of Lemma 5.1.** Assume first that  $r > 1$  and define  $f(x) = \rho^r(x^{1/r})$ . By means of approximation procedures in Constantin [6] we know that in order to prove (5.1) it will be enough to show that  $\forall x \geq 0$ ,  $f''(x) \leq 0$  holds true for each function  $\rho(\cdot) \in C^2(\mathbf{R}^+, \mathbf{R}^+)$  with  $\rho(0) = 0$ ,  $\rho'(x) \geq 0$  and  $\rho''(x) \leq 0$  for  $x \geq 0$ .

Indeed, we have  $f'(x) = \rho^{r-1}(x^{\frac{1}{r}}) \cdot x^{\frac{1}{r}-1} \cdot \rho'(x^{\frac{1}{r}})$ , and then

$$f''(x) = \frac{(r-1)t\rho^{r-2}(t)\rho'(t)}{rx^2} [\rho'(t)t - \rho(t)] + \frac{t^2\rho^{r-1}(t)\rho''(t)}{px^2}$$

with  $t = x^{1/r}$ . Considering that  $\rho(t) \geq 0$ ,  $\rho'(t) \geq 0$  and  $\rho''(t) \leq 0$ , it suffices to prove that  $\rho'(t)t - \rho(t) \leq 0$ . Note that Taylor's expansion yields that  $0 = \rho(0) = \rho(t) - t\rho'(t) + t^2\rho''(\xi_t)/2$  for  $t > 0$  and some  $\xi_t \in (0, t)$ . Since  $\rho''(\xi_t) \leq 0$ , the preceding relation proves  $\rho'(t)t - \rho(t) \leq 0$ . Then (5.1) is proved.

Next we prove (5.2). Let  $r < 1$ ,  $\rho(u) > 0$  for  $u > 0$ , and  $\int_{0+} \rho^{-1}(u) du = +\infty$ . Similar to (5.3) we know that  $\forall u \in [0, 1]$ ,  $\rho(u) \geq u\rho(1)$ . Consequently, we have

$$\int_{0+} \frac{u^{\frac{1-r}{r}} du}{r\rho(u^{\frac{1}{r}})} = \int_{0+} \frac{du}{\rho(u)} = +\infty$$

and

$$\liminf_{u \rightarrow 0+} \frac{\frac{1}{\rho^r(u^{\frac{1}{r}})}}{\frac{u^{\frac{1-r}{r}}}{\rho(u^{\frac{1}{r}})}} = \liminf_{u \rightarrow 0+} \left( \frac{\rho(u^{\frac{1}{r}})}{u^{\frac{1}{r}}} \right)^{1-r} \geq [\rho(1)]^{1-r} > 0,$$

from which (5.2) follows immediately. The proof of Lemma 5.1 is completed.  $\square$

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